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Abstract

In this paper we extend the conditions on *quasi-thermal-incompressible materials* presented in [1] so that they satisfy all the principles of thermodynamics, including the stability condition associated with the concavity of the chemical potential. We analyze the approximations under which a quasi-thermal-incompressible medium can be considered as incompressible. We find that the pressure cannot exceed a very large critical value and that the compressibility factor must be greater than a lower limit that is very small. The analysis is first done for the case of fluids and then extended to the case of thermoelastic solids.

Key words: Incompressible fluids and solids; Entropy principle; Chemical potential concavity.
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1 Introduction

It is well known that fully incompressible materials do not exist in nature. However, it is important to have a mathematical model for an incompressible medium, as an idealization of media that exhibit extreme resistance to volume

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change. In the context of isothermal mechanics, an ideal incompressible material is a medium that can only be deformed without any change in volume. Extensive literature has been devoted to qualitative analysis and numerical methods for constructing solutions of incompressible fluids as limits of compressible ones as the Mach number tends to zero (see for example [2,3]).
However, when the process is not isothermal, the notion of incompressibility is not well defined. Several possibilities arise.

For compressible fluids, the pressure is a constitutive function while for incompressible fluids the pressure is only a Lagrange multiplier associated with the constraint of incompressibility. Therefore, to compare the solutions of compressible and incompressible media, it is convenient to choose the pressure p (instead of the density ρ) and the temperature T as thermodynamic variables (see, e.g. [4,5]), the other quantities, such as specific volume $V = 1/\rho$ and internal energy ε, being determined by constitutive equations in the form:

$$V \equiv V(p,T)\,, \quad \varepsilon \equiv \varepsilon(p,T).$$

Two parameters are important for a fluid: the *thermal expansion coefficient* α and the *compressibility factor* β defined by

$$\alpha = \frac{V_T}{V}, \qquad \beta = -\frac{V_p}{V}\,, \tag{1}$$

where the subscripts T and p indicate partial derivatives with respect to variables T and p.
Experiments confirm that for fluids considered as incompressible the volume changes little with the temperature and remains practically unchanged with the pressure. For this reason, many authors consider as incompressible a material for which the specific volume does not vary with the pressure but varies only with the temperature (*i.e.*, $V \equiv V(T)$).

The first model of incompressibility was proposed by Müller [6]: Here, all the constitutive equations of an incompressible fluid do not depend on the pressure. We think that Müller's motivation stems from his using of variables ρ and T: in experiments, the density of incompressible materials depends on the temperature and it is reasonable to assume that this is the case for all constitutive functions as for instance for the internal energy.
Nevertheless, Müller proves that the only function $V \equiv V(T)$ compatible with the entropy principle is a constant. As previously indicated, this result obviously disagrees with experimental or theoretical results as in the so-called Boussinesq approximation (see, *e.g.* [7], [8]). We called this contradiction the *Müller paradox* [1].

A second, less restrictive, model usually employed in the literature requires that the only constitutive function independent of the pressure is the specific

volume (see, for example Rajagopal *et al* [4,5]) and as a result the Gibbs equation is satisfied. In a recent paper [1], we named such a material a *quasi-thermal-incompressible medium* and proved that for a pressure smaller than a critical value, the ideal medium of Müller can be recovered as a limit case.
Nonetheless, a weakness of $V \equiv V(T)$ as definition for incompressibility was first noted by Manacorda [9] who showed that instabilities occur in wave propagation. The instabilities are due to the non-concavity of the chemical potential and to the sound velocity c becoming imaginary; the mathematical system of Euler equations is of elliptic type. Let us also note that in the solid case, Dunwoody and Odgen [12] showed that conditions for infinitesimal stability are unattainable if the trace of the strain tensor is dependent on temperature. Starting from Manacorda's observation, other authors such as Scott *et al* [10,11] proposed an alternative definition of incompressibility: instead of $V \equiv V(T)$, they assumed $V \equiv V(S)$, where S is the specific entropy; then, in the ideal incompressible case, the Mach number becomes unbounded and the sound velocity becomes infinite. From an experimental point of view, this assumption is unrealistic because the entropy is not an observable and no direct evaluation of $V \equiv V(S)$ is possible. Moreover, this ideal limit case implies a parabolic structure for the Euler equations.

Due to the above considerations, the aim of our paper is to propose a more realistic model of incompressible medium as limit case of a quasi-thermal-incompressible material based on the following requirements:
(i) The model must respect all the principles of thermodynamics: both the Gibbs equation and the thermodynamic stability corresponding to the concavity of the chemical potential must be satisfied.
(ii) The model must fit with experiments when the compressibility factor is not zero but it is very small.
It is noteworthy that we obtain numerical results perfectly fitting with those obtained in paper [1]. The results are also extended to hyperelastic materials.

2 Thermodynamic restrictions

We consider the thermodynamic conditions verified by compressible fluids when the specific volume is governed by a constitutive function written in the form:

$$V \equiv V(p, T). \tag{2}$$

The entropy principle and the thermodynamic stability must be satisfied.

a) ***The entropy principle***

In local equilibrium, the entropy principle requires the validity of the Gibbs

equation:

$$TdS = d\varepsilon + p\,dV. \tag{3}$$

In addition for Navier-Stokes-Fourier fluids, the heat conductivity and the viscosity coefficients must be non negative although they are null for Euler fluids. The choice of independent variables p and T induces the chemical potential μ as a natural thermodynamic potential:

$$\mu = \varepsilon + p\,V - T\,S$$

and Eq. (3) is equivalent to

$$d\mu = V dp - S dT.$$

Other thermodynamic variables derive from the chemical potential:

$$V = \mu_p\,, \qquad S = -\mu_T \tag{4}$$

and the specific internal energy is

$$\varepsilon = \mu - p\ \mu_p - T\ \mu_T, \tag{5}$$

where for any function $f \equiv f(p,T)$, we denote

$$f_p = \left(\frac{\partial f}{\partial p}\right)_T, \quad f_T = \left(\frac{\partial f}{\partial T}\right)_p.$$

The thermal equation of state (2) is determined by experiments while the chemical potential and the entropy density can be deduced as follows:

$$\mu = \int V(p,T)\,dp + \widetilde{\mu}\,(T), \qquad S = -\int V_T(p,T)\,dp - \widetilde{\mu}'\,(T), \tag{6}$$

where $\widetilde{\mu}\,(T)$ is a function only depending on T. From Eq. (5) we get

$$\varepsilon = e(T) + \int V\,dp - p\,V - T\int V_{_T}\,dp, \tag{7}$$

with

$$e(T) \equiv\ \widetilde{\mu}\,(T) - T\,\widetilde{\mu}\,'(T) \quad \text{or equivalently} \quad \widetilde{\mu}(T) = -T\int \frac{e(T)}{T^2}\,dT. \tag{8}$$

We summarize this as a statement:

Statement 1 - *For any constitutive functions $V \equiv V(p,T)$ and $e \equiv e(T)$, the entropy principle is satisfied if the chemical potential, entropy density and internal energy are given by Eq.* $(6)_1$*, Eq.* $(6)_2$ *and* (7)*, together with Eq.* $(8)_2$*.*

b) ***Thermodynamic stability***

The specific heat $\mathrm{C_P}$ is defined as the partial derivative of the specific enthalpy $h \equiv \varepsilon + p\,V$ with respect to T at constant pressure p. Consequently, Eq. (7) yields

$$\mathrm{C_P} = e'(T) - T \int V_{TT}\, dp\ . \tag{9}$$

Thermodynamic stability requires that the chemical potential be a concave function of p and T:

$$\mu_{pp} = V_p < 0, \qquad \left(\mu_{TT} = -\frac{\mathrm{C_P}}{T} < 0\right) \tag{10}$$

and

$$I \equiv \mu_{TT}\,\mu_{pp} - \mu_{Tp}^2 = -\frac{\mathrm{C_P}\,V_p}{T} - V_T^2 > 0 \quad \Longleftrightarrow \quad V_p < -\frac{T\,V_T^2}{\mathrm{C_P}}. \tag{11}$$

By using Eq. (1), inequality (11) can be written in terms of the thermal expansion coefficient α and the compressibility factor β:

$$\beta > \beta_{cr}, \qquad \beta_{cr} = \frac{\alpha^2 TV}{\mathrm{C_P}} > 0. \tag{12}$$

Statement 2 - *Thermodynamic stability requires that the state functions* $V \equiv V(p,T)$ *and* $e \equiv e(T)$ *satisfy the inequalities:*

$$V_p < -\frac{T\,V_T^2}{\mathrm{C_P}}, \qquad \mathrm{C_P} > 0.$$

Consequently, there exists a lower bound limit β_{cr} *of* β *such that if* $\beta > \beta_{cr}$*, then the material is stable.*

The adiabatic sound velocity c is:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_S = -V^2 \left(\frac{\partial p}{\partial V}\right)_S .$$

From Eq. (4) we obtain

$$dV = \mu_{pT}\, dT + \mu_{pp}\, dp, \qquad dS = -\mu_{TT}\, dT - \mu_{Tp}\, dp. \tag{13}$$

When $dS = 0$, Eq. $(13)_2$ substituted in Eq. $(13)_1$ yields p as a function of V and we get:

$$c^2 = -\frac{\mu_p^2\,\mu_{TT}}{I}. \tag{14}$$

Therefore, when the chemical potential is a concave function of p and T, we automatically get $c^2 > 0$ and the differential system for Euler fluids is hyperbolic. Taking account of Eqs. $(6)_1$, $(8)_2$, (9), (11) and (12), Eq. (14)

yields the sound velocity in the form:

$$c = \sqrt{\frac{V}{\beta - \beta_{cr}}}\,.$$

We observe that when $V \equiv V(T)$ the model of incompressibility corresponds to $\beta = 0$ and for Euler fluids the differential system is elliptic; when $V \equiv V(S)$, the model of incompressibility corresponds to $\beta \equiv \beta_{cr}$ and the system is parabolic. In our case $\beta > \beta_{cr}$, the differential system is hyperbolic and the fluid is stable.

3 Quasi-Thermal-Incompressible Materials

For the so-called *incompressible fluids*, the volume changes little with the temperature and changes very little with the pressure. Experiments confirm this assumption.
In the neighborhood of a reference state (p_0, T_0, V_0), we choose a small dimensionless parameter δ ($\delta \ll 1$) such that:

$$\delta = \alpha_0\, T_0, \tag{15}$$

and moreover, we assume that β_0 is of order δ^2:

$$\beta_0\, p_0 = O(\delta^2), \tag{16}$$

where α_0 and β_0 are the thermal expansion coefficient and the compressibility factor at the reference state.

Definition 1 - *A compressible fluid satisfying the thermodynamic conditions of Section 2 is called an Extended-Quasi-Thermal-Incompressible fluid (EQTI) if there exist $\widehat{V}(T)$ and $\widehat{\varepsilon}(T)$ such that*

$$V(p,T) = \widehat{V}(T) + O(\delta^2) \text{ with } \widehat{V}'(T) = O(\delta) \text{ and } \varepsilon(p,T) = \widehat{\varepsilon}(T) + O(\delta^2). \tag{17}$$

This means that an *EQTI* material is a stable compressible fluid that approximates an incompressible fluid to order δ^2 in the sense of Müller's definition.
Conditions (15)-(16) together with Eq. $(17)_1$ yield the representation of $V(p,T)$:

$$V(p,T) = V_0 + \delta\, W(T) - \delta^2 U(p,T), \tag{18}$$

where $W(T)$ and $U(p,T)$ are two constitutive functions chosen in agreement with conditions in Section 2. From Eqs. $(6)_1$ and (18) we deduce:

$$\mu = \widetilde{\mu}\,(T) + p\, V_0 + \delta\, p\; W(T) + \delta^2 \widehat{\mu}(p,T), \quad \widehat{\mu}(p,T) = -\int U(p,T)\; dp.$$

Since we can incorporate its limit value in $\widetilde{\mu}\,(T)$ without loss of generality, we can chose the function $\widehat{\mu}$ so that $\lim_{p->0}\widehat{\mu}(p,T)=0$. From Eq. (7) we obtain:

$$\varepsilon(p,T)=e(T)-\delta\,T\,W'(T)\;p+O(\delta^2). \tag{19}$$

Due to Eq. (19), in order to satisfy Eqs. $(17)_2$ we require that the pressure cannot exceed a critical value:

$$p\ll p_{cr}(T)\qquad\text{with}\qquad p_{cr}(T)=\frac{1}{\delta}\frac{e(T)}{T\,W'(T)}. \tag{20}$$

We observe that the critical value is large and of order δ^{-1}. In incompressible materials, the pressure cannot exceed a critical value depending on temperature. Inequality (20) implies

$$\varepsilon(p,T)=e(T)+O(\delta^2). \tag{21}$$

Moreover, from Eqs. (1) and (15) we obtain

$$\alpha=\delta\;\frac{W'(T)}{V_{_0}}+O(\delta^2),\quad W'(T_{_0})=\;\frac{V_{_0}}{T_{_0}};\qquad\beta=\delta^2\;\frac{U_p(p,T)}{V_{_0}}.$$

From Eqs. (9) and (12) we get

$$\beta_{cr}=\delta^2\,\frac{T\,W'^2(T)}{V_{_0}\,\mathrm{C_P}},\qquad \mathrm{C_P}\simeq e'(T),$$

and the thermodynamic stability is ensured when

$$U_p(p,T)>\frac{T\,W'^2(T)}{\mathrm{C_P}},\qquad \mathrm{C_P}\simeq e'(T)>0. \tag{22}$$

Finally, from Eq. (14) we deduce that the dominant part in sound velocity is of order δ^{-1}:

$$c\simeq\frac{V_{_0}}{\delta}\sqrt{\frac{\mathrm{C_P}}{U_p\,\mathrm{C_P}-TW'^2}}\;. \tag{23}$$

We can conclude with

Statement 3 - *An EQTI fluid given by constitutive functions* (18) *and* (21)*, satisfying inequality* (22) *is a good approximation of an incompressible fluid as* V *and* ε *differ to order* δ^2 *from functions depending only on* T*, provided the pressure is smaller than a critical pressure* p_{cr} *given by Eq.* (20)*. The sound velocity given by Eq.* (23) *is real.*

Remark: We notice that in the limit case of isothermal processes for which $W(T)\equiv 0$ and $U\equiv U(p)$, we have $\beta_{cr}\equiv 0$ and inequality $(22)_1$ and the sound velocity become $U'(p)>0$ and $c=V_0/(\delta\sqrt{U'})$, respectively. In this case again, the *EQTI* requires that V is not constant but is function of p in the form $V=V_0-\delta^2U(p)$.

3.1 Linear dependence of V with respect to T and p

The most significant case is the linear expansion of V near (T_0, p_0):

$$V = V_0\{1 + \alpha\,(T - T_0) - \beta\,(p - p_0)\} \quad \text{with} \quad e = \mathrm{C_P}\,T\,. \tag{24}$$

In this case, the scalars α, β and $\mathrm{C_P}$ are positive constants. Expression (24) is a particular case of Eq. (18) under the identifications:

$$\alpha\,T_0 = \delta, \quad W(T) = \frac{V_0}{T_0}\,(T - T_0), \quad U(p,T) = \frac{\beta}{\alpha^2}\frac{V_0}{T_0^2}\,(p - p_0). \tag{25}$$

Then, the fluid is *EQTI* if

$$\alpha\,T_0 = \delta \ll 1 \quad \text{and} \quad \beta > \beta_{cr} \quad \text{with} \quad \beta_{cr} = \delta^2\frac{V_0}{\mathrm{C_P}\,T_0}, \tag{26}$$

together with

$$p \ll p_{cr}, \qquad \text{with} \qquad p_{cr} = \frac{1}{\delta}\frac{\mathrm{C_P}\,T_0}{V_0}. \tag{27}$$

Relation (23) and relation (25) yield the adiabatic sound velocity:

$$c_0 = \sqrt{\frac{\mathrm{C_P}\,V_0}{\beta\,\mathrm{C_P} - \alpha^2 V_0 T_0}} \qquad \text{or} \qquad \beta = \frac{V_0}{c_0^2} + \alpha^2\,\frac{T_0 V_0}{\mathrm{C_P}}\,. \tag{28}$$

Consequently, a fluid can be considered as incompressible if the pressure is smaller than a critical pressure which is of order α^{-1}. This is similar to conclusions in paper [1]. In addition, to ensure the convexity of the chemical potential, the compressibility factor must be very small but not identically null: β must be greater than a critical value β_{cr} which is of order α^2. From Eq. (28), the value of the sound velocity allows to calculate the value of β. We finally observe that, in the Boussinesq approximation, $\beta \neq 0$ induces an additional term depending on p in the density expansion near (T_0, p_0):

$$\rho = \rho_0\{1 - \alpha\,(T - T_0) + \beta\,(p - p_0)\}.$$

Therefore, these considerations should be useful to revisit and to justify more rigorously the Boussinesq approximation. We point out that in this case we obtain constitutive equations that may be implicit in the direction pointed out in [13].

3.2 Case of liquid water

Numerical values for liquid water are obtained in [14]:

$p_0 = 10^5\,\mathtt{Pascal};\ V_0 = 10^{-3}\,\mathtt{m}^3/\mathtt{kg};\ T_0 = 293\ {}^\circ\mathtt{K};\ \mathrm{C_P} = 4.2 \times 10^3\mathtt{Joule}/\mathtt{kg}\ {}^\circ\mathtt{K};$

$$c = 1420\ \mathtt{m/sec}\,;\ \alpha = 2.07 \times 10^{-4}/\,^{\circ}\mathtt{K};\ \beta_{cr} = 3 \times 10^{-12}/\,\mathtt{Pascal}.$$

Inequality (27) reads

$$p \ll p_{cr} = 2 \times 10^{10}\,\mathtt{Pascal} = 2 \times 10^{5}\,\mathtt{atm},$$

which is the same critical pressure as in paper [1]. We observe that Eq. (28) yields $\beta = 4.98 \times 10^{-10}/\,\mathtt{Pascal}$ which automatically satisfies the inequality (26). Liquid water is indeed a very good example of an *EQTI* liquid.

4 Quasi-thermal-incompressibility for hyperelastic media

4.1 Generalities

As done in paper [1] and by taking up the definition of pressure presented by Flory in 1953 for rubber gum [15,16] and extended for hyperelastic media by Gouin and Debiève in 1986 [17] and Rubin in 1988 [18], we can extend the results from fluids to thermo-elastic materials. With this aim we define

$$\widetilde{\boldsymbol{C}} = \frac{1}{(\det \boldsymbol{C})^{\frac{1}{3}}}\,\boldsymbol{C} \qquad \text{or} \qquad \boldsymbol{C} = J^{\frac{2}{3}}\,\widetilde{\boldsymbol{C}}\,, \tag{29}$$

where $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$ is the right Cauchy stress deformation tensor, $\boldsymbol{F}$ is the deformation gradient, $J = \det \boldsymbol{F} = \rho_0/\rho$ and ρ_0 is the reference density. The specific free energy can be expressed in the form:

$$\psi \equiv f(\rho, \widetilde{\boldsymbol{C}}, T)\,,$$

where the independent variables ρ and $\widetilde{\boldsymbol{C}}$ are used instead of $\boldsymbol{C}$. Since $\det \widetilde{\boldsymbol{C}}$ =1, the variable ρ corresponds to the change of volume, while the tensorial variable $\widetilde{\boldsymbol{C}}$ is associated with the *distortion* of the medium: we call $\widetilde{\boldsymbol{C}}$ *the pure deformation of the hyperelastic body.* This point is fundamental for the decomposition of the stress tensor. If f is independent of $\widetilde{\boldsymbol{C}}$, then we are back to the fluid case. It is convenient to introduce the function g such that:

$$g(\rho, \boldsymbol{C}, T) \equiv f\left(\rho, \frac{1}{(\det \boldsymbol{C})^{\frac{1}{3}}}\,\boldsymbol{C}, T\right).$$

Consequently, the free energy of an hyperelastic material can be defined as $\psi \equiv g(\rho, \boldsymbol{C}, T)$ where g is a homogeneous function of degree zero with respect to $\boldsymbol{C}$. We deduce the Cauchy stress tensor of the medium in the form [17]:

$$\boldsymbol{t} = -p\,\boldsymbol{1} + \ \boldsymbol{\tau}\,, \quad \text{with} \quad p = \rho^2\frac{\partial g}{\partial \rho}\,, \quad \boldsymbol{\tau} = 2\rho\,\boldsymbol{F}\frac{\partial g}{\partial \boldsymbol{C}}\,\boldsymbol{F}^T \text{ and } \ \operatorname{tr}\boldsymbol{\tau} = 0\,, \tag{30}$$

where tr is the trace operator. If g is independent of $\boldsymbol{C}$, p corresponds to the thermodynamic pressure. As proved in [17], p must be considered as the *pressure of the hyperelastic material*. Let us note that the pressure and the change of volume are observable: they can be experimentally measured by using a spherical elastic test-apparatus submitted to isotropic stresses.
The Gibbs equation in the case of elastic materials is [19,20]:

$$T dS = d\varepsilon - \frac{1}{2\rho_0}\, \boldsymbol{S} \cdot d\boldsymbol{C}\,, \tag{31}$$

where the *dot* represents the scalar product between matrices and $\boldsymbol{S} = J\ \boldsymbol{F}^{-1}\,\boldsymbol{t}\,(\boldsymbol{F}^T)^{-1}$ is the second Piola-Kirchhoff stress tensor. By inserting Eq. $(30)_1$ into $\boldsymbol{S}$, Eq. (31) yields the Gibbs relation:

$$T dS = d\varepsilon + p\ dV - J^{-\frac{2}{3}}\ \widetilde{\boldsymbol{\tau}} \cdot d\boldsymbol{C}, \tag{32}$$

with $\widetilde{\boldsymbol{\tau}}$ given by:

$$\widetilde{\boldsymbol{\tau}} = \frac{1}{2\,\rho}\, J^{\frac{2}{3}}\ \boldsymbol{F}^{-1}\boldsymbol{\tau}\boldsymbol{F}^{-1^T}.$$

If we take account of Eq. (29),

$$d\boldsymbol{C} = \frac{2}{3}\ \rho\ \widetilde{\boldsymbol{C}}\ dV + J^{\frac{2}{3}}\ d\widetilde{\boldsymbol{C}},$$

and that

$$\widetilde{\boldsymbol{\tau}} \cdot \widetilde{\boldsymbol{C}} = \frac{1}{2\,\rho}\boldsymbol{F}^{-1}\boldsymbol{\tau}\boldsymbol{F}^{-1^T}.\boldsymbol{C} = \frac{1}{2\,\rho}\,\mathtt{tr}\left(\boldsymbol{F}^{-1}\boldsymbol{\tau}\boldsymbol{F}^{-1^T}\boldsymbol{F}^T\boldsymbol{F}\right) \equiv \frac{1}{2\,\rho}\,\mathtt{tr}\,\boldsymbol{\tau} = 0, \tag{33}$$

we obtain the Gibbs relation (32) in the final form

$$T dS = d\varepsilon + p\ dV - \widetilde{\boldsymbol{\tau}} \cdot\ d\widetilde{\boldsymbol{C}}\,. \tag{34}$$

As in the fluid case, we introduce the chemical potential $\mu = \varepsilon + p\,V - TS - \widetilde{\boldsymbol{\tau}} \cdot \widetilde{\boldsymbol{C}}$. Thanks to the orthogonality condition (33), μ takes the same form as for fluids:

$$\mu = \varepsilon + p\,V - TS.$$

Equation (34) implies

$$d\mu = V dp - S dT + \widetilde{\boldsymbol{\tau}} \cdot\ d\widetilde{\boldsymbol{C}}. \tag{35}$$

Consequently, the change of variables from (V, T) into (p, T) is natural and the variable $\widetilde{\boldsymbol{C}}$ does not change. Equation (35) implies:

$$V = \mu_p\,, \quad S = -\mu_{_T}\,, \quad \widetilde{\boldsymbol{\tau}} = \mu_{\widetilde{\boldsymbol{C}}}\,, \quad \varepsilon = \mu - T\mu_{_T} - p\,\mu_p\,, \tag{36}$$

with

$$\mu_p = \left(\frac{\partial \mu}{\partial p}\right)_{T,\,\widetilde{\boldsymbol{C}}}, \quad \mu_T = \left(\frac{\partial \mu}{\partial T}\right)_{p,\,\widetilde{\boldsymbol{C}}}, \quad \mu_{\widetilde{\boldsymbol{C}}} = \left(\frac{\partial \mu}{\partial \widetilde{\boldsymbol{C}}}\right)_{p,\,T}.$$

Let us assume that the specific volume is written as a function of $p, T, \widetilde{\boldsymbol{C}}$:

$$V \equiv V(p, T, \widetilde{\boldsymbol{C}}).$$

By integration of Eq. $(36)_1$, we obtain

$$\mu = \int V(p, T, \widetilde{\boldsymbol{C}})\, dp + \widehat{\mu}(T, \widetilde{\boldsymbol{C}}), \quad S = -\int V_T(p, T, \widetilde{\boldsymbol{C}})\, dp - \widehat{\mu}_T(T, \widetilde{\boldsymbol{C}})\,, \quad (37)$$

where $\widehat{\mu}$ is an additive function depending only on T and $\widetilde{\boldsymbol{C}}$. By substituting in Eq. $(36)_4$, we get:

$$\varepsilon = e(T, \widetilde{\boldsymbol{C}}) + \int V(p, T, \widetilde{\boldsymbol{C}})\, dp - p\, V - T \int V_{_T}\, dp\,, \quad (38)$$

with the additional function

$$e(T, \widetilde{\boldsymbol{C}}) = \widehat{\mu} - T\widehat{\mu}_T \text{ or equivalently } \widehat{\mu}(T, \widetilde{\boldsymbol{C}}) = -T \int \frac{e(T, \widetilde{\boldsymbol{C}})}{T^2}\, dT + k(\widetilde{\boldsymbol{C}}), \quad (39)$$

where $k(\widetilde{\boldsymbol{C}})$ is an arbitrary function of $\widetilde{\boldsymbol{C}}$. The additional function is in the same form than for fluids. Therefore

Statement 4 - *For any constitutive functions* $V \equiv V(p, T, \widetilde{\boldsymbol{C}})$, *and* $e \equiv e(T, \widetilde{\boldsymbol{C}})$, *the entropy principle is satisfied if the chemical potential, the entropy density and the internal energy are given by Eqs.* $(37)_1$, $(37)_2$, (38) *with Eq.* $(39)_2$, *respectively.*

In nonlinear theories, the concavity of the entropy density may be valid only in some domain of state variables. In particular this is the case in nonlinear elasticity where the concavity is in contradiction with the objectivity principle if the deformation is large [19] (see also [21,22]). The concavity of the chemical potential is valid only for sufficient small deformations in the neighborhood of undeformed configuration. In the decomposition of the stress tensor given by Eq. (30), the pressure p is the main observable variable and the pure deformation $\widetilde{\boldsymbol{C}}$ does not affect the change of variables between internal energy and chemical potential. Consequently, taking $\widetilde{\boldsymbol{C}}$ constant we conclude that inequalities (10-11) are necessary conditions for stability.

4.2 Quasi-Thermal-Incompressible Solids

In experiments, the specific volume of incompressible solids changes if the temperature varies. In the literature it is usually assumed that $J \equiv J(T)$ (or equivalently $V \equiv V(T)$). Therefore, it is natural to define *EQTI* solids similarly to the definition given for fluids.

Definition 2 - *A compressible solid satisfying thermodynamic conditions associated with the entropy principle and stability is called an Extended-Quasi-Thermal-Incompressible solid if there exist* $\widehat{V}(T)$ *and* $\widehat{\varepsilon}(T,\widetilde{\boldsymbol{C}})$ *such that*

$$V(p,T,\widetilde{\boldsymbol{C}}) = \widehat{V}(T)+O(\delta^2) \text{ with } \widehat{V}_T = O(\delta) \quad \text{and} \quad \varepsilon(p,T,\widetilde{\boldsymbol{C}}) = \widehat{\varepsilon}(T,\widetilde{\boldsymbol{C}})+O(\delta^2).$$

Assuming for solids the linear expansion (24) for V, we derive a similar result as for fluids:

Statement 5 - *A thermoelastic material with constitutive equation* (24) *is a stable EQTI and tends to an incompressible material under the conditions*

$$\beta > \beta_{cr} \quad \text{with} \quad \beta_{cr} = \delta^2 \frac{V_0}{\mathrm{C_P} T_0}, \qquad \text{and} \qquad p \ll p_{cr}, \quad \text{with} \quad p_{cr} = \frac{1}{\delta}\frac{\mathrm{C_P} T_0}{V_0},$$

where V_0 *and* $\delta = \alpha T_0 \ll 1$ *are positive constants, while* C_P *may depend on the pure deformation* $\widetilde{\boldsymbol{C}}$.

4.3 Case of pure gum rubber

As in paper [1], we consider the case of rubber as example of hyperelastic material. To verify the conditions *EQTI,* we get experimental values from the literature: numerical values for rubber are obtained in [14,23]:

$$p_0 = 10^5\,\mathtt{Pascal};\ \mathrm{C_P} = 1.9\times 10^3 \mathtt{Joule/kg}\ {}^\circ\mathtt{K};\ V_0 = 1.08\times 10^{-3}\mathtt{m}^3/\mathtt{kg};$$

$$T_0 = 273\ {}^\circ\mathtt{K};\ c = 54\mathtt{m/sec};\ \alpha = 7\times 10^{-3}/\ {}^\circ\mathtt{K};\ \beta = 4.5\times 10^{-7}/\,\mathtt{Pascal};$$

$$\beta_{cr} = 7.6\times 10^{-9}/\,\mathtt{Pascal};\ p_{cr} = 2.5\times 10^8\,\mathtt{Pascal} = 2.5\times 10^3\,\mathtt{atm}.$$

We can immediately verify that rubber is also a good example of an *EQTI* medium. Let us note that the considered values for rubber are a little different from those in [1]; this is due to the fact that the sound velocity was measured at 273°K and not at 325°K, as in [1].

5 Conclusion

Our goal was to present a model expressing the limit case of incompressible materials and fitting with the principles and conditions of thermodynamics. Earlier models were thermodynamically deficient. Here, we present a new model (which we call *Extended-Quasi-Thermal-Incompressible* model) starting from two simple requirements:
(i) An incompressible medium does not physically exist but it is the limit case

of a compressible medium that verifies all thermodynamic conditions.
(ii) On physical grounds, the volume changes little with the temperature and does not practically change with the pressure.
As result, the conditions for the pressure are the same than in previous paper [1] but an additional condition associated with the compressibility factor must be verified. We point out that a decomposition of the stress tensor for hyperelastic media into a pressure term associated with the *change of volume* and a null-trace part associated with the *pure deformation* of the medium leads to analogous conditions of *EQTI* for fluids and hyperelastic materials. Let us note that on a phenomenological basis a theory where the elasticity parameters depend on the pressure has been developed in [24,25].

From a mathematical standpoint, the difficult question of proving that the solutions depend continuously on δ and that the limit exists as δ tends to zero, remains an open problem.

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References

[1] Gouin H., Muracchini A., Ruggeri T., *On the Müller paradox for thermal incompressible media*, *on line in* Continuum Mechanics and Thermodynamics (2011), DOI:10.1007/s00161-011-0201-1.

[2] Klainerman S., Majda S., *Compressible and incompressible fluids*, Communications in Pure and Applied Mathematics, 35, 629-653 (1982).

[3] Lions P.L., Masmoudi N., *Incompressible limit for a viscous compressible fluid*, Journal de Mathématiques Pures et Appliquées, 77, 585-627 (1998).

[4] Rajagopal K.R., Ruzika M., Srinivasa A.R., *On the Oberbeck-Boussinesq approximation*, Mathematical Models and Methods in Applied Sciences, 6, 1157-1167 (1996).

[5] Rajagopal K.R., Saccomandi G., Vergori L., *On the Oberbeck-Boussinesq approximation for fluids with pressure dependent viscosities*, Nonlinear Analysis: Real World Applications, 10, 1139-1150 (2009).

[6] Müller I., *Thermodynamics*, Pitman, London (1985).

[7] Chandrasekhar S., *Hydrodynamic and hydromagnetic stability*, ch. 2, sect. 8, p. 16, Dover New York (1981) (reedited from Oxford University Press 1961).

[8] Straughan B., *The energy method, stability and nonlinear convection*, ch 3, sect. 3.2, pp. 48-49, 2nd Ed., Springer (2004).

[9] Manacorda T., *Onde elementari nella termoelasticità di solidi incomprimibili*, Atti Accademia Scienze Torino, pp. 503-509 (1967).

[10] Scott N.H., *Linear dynamical stability in constrained thermoelasticity: II. Deformation-entropy constraints*, The Quarterly Journal of Mechanics and Applied Mathematics, 45, 651-662 (1992).

[11] Scott N.H., *Thermoelasticity with thermomechanical constraints*, International Journal of Non-Linear Mechanics, 36, 549-564 (2001).

[12] Dunwoody J., Ogden R.W., *On the thermodynamic stability of elastic heat-conducting solids subject to a deformation-temperature constraint*, Mathematics and Mechanics of Solids, 7, 285-306 (2002).

[13] Rajagopal K.R., *On implicit constitutive theories*, Applications of Mathematics, 48, 279-319 (2003).

[14] Weast R.C., Astle M.J., Beyer W.H., Eds., *CRC Handbook of Chemistry and Physics*, CRC Press, Boca Raton (1988).

[15] Flory P.J., *Principles of polymer chemistry*. Cornell University Press, Ithaca, New York (1953).

[16] Flory P.J., *Thermodynamic relations for high elastic materials*, Transactions of the Faraday Society, 57, 829-838 (1961).

[17] Gouin H., Debiève J.F., *Variational principle involving the stress tensor in elastodynamics*, International Journal of Engineering Science, 24, 1057-1066 (1986), http://arXiv:0807.3454.

[18] Rubin M.B., *The significance of pure measures of distorsion in nonlinear elasticity with reference to the Poynting problem*, Journal of Elasticity, 20, 53-64 (1988).

[19] Truesdell C., Toupin R.A., *Principles of classical mechanics and field theory*, Encyclopedia of Physics, Vol.III/1, Springer, Berlin (1960).

[20] Ruggeri T., *Introduzione alla termomeccanica dei continui*. Monduzzi, Bologna (2007).

[21] Dafermos C., *Hyperbolic conservation laws in continuum physics*, Springer, Berlin (2001).

[22] Ruggeri T., Sugiyama M., *Hyperbolicity, convexity and shock waves in one-dimensional crystalline solids*, Journal of Physics A: Mathematical and General, 38, 4337-4347 (2005).

[23] http://physics.info/expansion/ ; http://www.matbase.com/

[24] Rajagopal K.R., Srinivasa A.R., *On a class of non-dissipative materials that are not hyperelastic*, Proceedings of the Royal Society A, 465, 493-500 (2009).

[25] Rajagopal K.R., Saccomandi G., *The mechanics and mathematics of the effect of pressure on the shear modulus of elastomers*, Proceedings of the Royal Society A, 465, 3859-3874 (2009).